MOTION OF A VORTEX PAIR BETWEEN PARALLEL WALLS

P. I. Geshev and B. S. Ezdin

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1. INTRODUCTION

In addition to ideal fluid hydrodynamics [1, 2], the concept of point vortices is used in the theory of superfluid helium [3], as well as in the model of a strongly magnetized plasma where the behavior of the electrons and ions is mathematically equivalent to vortex motion [4]. Although energy dissipation effects are always present in real phenomena occurring in a continuous medium, a large class of problems exists, where these effects can be neglected. This is valid for any vortex system being examined in short time intervals. In liquid helium at temperatures close to absolute zero, the energy dissipation can generally be neglected, and the system of quantum vortices considered as ideal [3]. The question of integrability of the vortex equations of motion is interesting. The motion in n-dimensional phase space occurs along a manifold of dimensionality $n - k$, where k is the number of integrals of the dynamical system. The necessary condition of stochasticity of the motion is $n - k > 2$ [5]. It is shown in [6, 7] that the problem of three vortices in an unlimited space is integrated exactly while stochastic trajectories [6, 8] appear in a system of four vortices under definite initial conditions. If the domain of vortex motion is bounded by impermeable walls, then the stochasticity already appears in a system of two vortices [9, 10].

It is shown in this paper that the motion of two vortices between parallel walls is still another exactly integrable case.

2. EQUATIONS OF MOTION AND CONSERVATION INTEGRALS

Kirchhoff [11] first obtained the equations of motion for a system of vortices in unlimited space. Routh [12] described the motion of one vortex in a simply connected domain, and Lin proposed the solution for a system of N vortices in a multiconnected domain [13]. In the case of the motion of N vortices, the method of conformal mapping [10] is used in a simply-connected domain.

Let the vortex motion occur in a two-dimensional simply-connected domain with the known boundary Γ. In this case all the hydrodynamic information can be obtained from the stream function ψ that satisfies the Poisson equation

$$\Delta\psi = -\omega \tag{2.1}$$

and is the third component of the vector potential $A = (0, 0, \psi)$ of the velocity field $u = \mathrm{rot}\, A$. The right side in (2.1) is the vorticity of the velocity field. In the case of point ideal vortices ω can be written in the form

$$\omega = \sum_{n=1}^{N} \gamma_n \delta(\mathbf{x} - \mathbf{x}_n(t)), \tag{2.2}$$

where γ_n is the circulation around the n-th vortex, $\delta(\mathbf{x} - \mathbf{x}_n(t))$ is a delta function, and $x_n(t)$ is the instantaneous location of the n-th vortex. In order to solve (2.1) with the right side (2.2) and the following condition on the boundary

$$\psi(\mathbf{x}_n \in \Gamma) = 0, \tag{2.3}$$

it is necessary to known an analytic function mapping the domain of the real vortex motion D of the $z = x + iy$ plane onto the canonical domain $|w| \leqslant 1$, a circle in the $w = u + iv$ plane. Then the Green's function of the problem (2.1)-(2.3) is determined by the formula [14]

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{1}{2\pi} \ln \left| \frac{w(z) - w(z_0)}{1 - w(z)\overline{w(z_0)}} \right|.$$

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The stream function has the form

$$\psi(\mathbf{x}) = \left|\sum_{n=1}^{N} \frac{\gamma_n}{2\pi} \ln\left|\frac{w(z) - w(z_n)}{1 - w(z)\overline{w(z_n)}}\right|\right|, \tag{2.4}$$

where the bar denotes the complex conjugate. The velocity of the k-th vortex is determined by all the system vortices and their images in the walls. The k-th vortex does not induce velocity upon itself. Consequently, to find the velocity of the k-th vortex from (2.4), the term $(\gamma_k/2\pi)\ln|z - z_k|$ should be subtracted, and then the derivatives of the difference being obtained at the point $z = z_k$ should be taken according to the definition of the velocity. Then the complex vortex velocity is determined by the equation

$$\frac{d\bar{z}_k}{dt} = \frac{\gamma_k w_k''}{4\pi i w_k'} - \frac{\gamma_k}{2\pi i}\frac{w_k'}{w_k - \bar{w}_k^{-1}} + w_k' \sum_{n \neq k}^{N} \frac{\gamma_n}{2\pi i}\left[\frac{1}{w_k - w_n} - \frac{1}{w_k - \bar{w}_n^{-1}}\right], \tag{2.5}$$

where $w_k = w(z_k)$; $w_k' = dw(z_k)/dz_k$; $w_k'' = d^2 w(z_k)/dz_k^2$. For $k = 1, \ldots, N$ the formulas (2.5) yield the dynamic equations of motion of a system of N point vortices.

The Hamiltonian of a system of N ideal vortices is obtained in [10]:

$$H(z_k, \bar{z}_k) = -\frac{1}{4\pi}\sum_{k=1}^{N}\left[\gamma_k^2 \ln\left|\frac{w_k'}{1 - w_k\bar{w}_k}\right| + \gamma_k \sum_{n \neq k}^{N} \gamma_n \ln\left|\frac{w_k - w_n}{1 - w_k\bar{w}_n}\right|\right]. \tag{2.6}$$

It is easy to see that the system of equations (2.5) can be represented in the Hamiltonian form

$$\gamma_k \frac{d\bar{z}_k}{dt} = 2i\frac{\partial H(z_k, \bar{z}_k)}{\partial z_k}$$

or

$$\gamma_k \dot{x}_k = \partial H(x_k, y_k)/\partial y_k, \quad \gamma_k \dot{y}_k = -\partial H(x_k, y_k)/\partial x_k. \tag{2.7}$$

The canonically conjugate variables are $q_k = x_k$, $p_k = \gamma_k y_k$.

If the physical domain of motion D is a circle ($w \equiv z$), because of invariance of the Hamiltonian (2.6) relative to rotation, an additional momentum integral exists $M = \sum_{k=1}^{N} \gamma_k |z_k|^2 = \text{const}$. The constancy of M is related to conservation of the total moment of fluid momentum in the circle [10]. If the "physical" domain of vortex motion is a strip, then the Hamiltonian (2.6) is invariant relative to shift along it, and therefore, an additional motion integral, the "impulse", should exist. Indeed, for a shift in the coordinate system by δx, the vortex coordinates receive the increment $\delta z_k = \delta x$. Evaluating the variation H by using (2.7), and equating it to zero, we obtain

$$\delta H = \sum_{k=1}^{N} \frac{\partial H}{\partial x_k}\delta x_k + \frac{\partial H}{\partial y_k}\delta y_k = \frac{d}{dt}\left(\sum_{k=1}^{N} \gamma_k y_k\right)\delta x,$$

from whence it is seen that the quantity ("impulse") $P = \sum_{k=1}^{N} \gamma_k y_k$ is conserved.

3. MOTION OF VORTICES IN THE STRIP $|y| \leqslant 0.5$, $|x| < \infty$

Formulas for the direct and reverse mapping of a strip on a unit circle have the form [15]

$$w = \frac{\exp(\pi z) - 1}{\exp(\pi z) + 1}, \quad z = \frac{1}{\pi}\ln\frac{1 + w}{1 - w}. \tag{3.1}$$

Using (3.1) and (2.6), we obtain the Hamiltonian for a system of two vortices in a strip

$$H = -\frac{1}{4\pi}\left\{\gamma_1^2 \ln\left|\frac{\pi e^{\pi z_1}}{e^{\pi z_1}+e^{\pi \bar{z}_1}}\right| + \gamma_2^2 \ln\left|\frac{\pi e^{\pi z_2}}{e^{\pi z_2}+e^{\pi \bar{z}_2}}\right| + 2\gamma_1\gamma_2 \ln\left|\frac{e^{\pi z_1}-e^{\pi z_2}}{e^{\pi z_1}+e^{\pi \bar{z}_2}}\right|\right\}.$$

The Hamiltonian of (2.7) is written in the form

$$\dot{x}_1 = -\frac{\gamma_1}{4}\,\mathrm{tg}\,\pi y_1 - \frac{\gamma_2}{4}\left[\frac{\sin\pi(y_1-y_2)}{\mathrm{ch}\,\pi(x_1-x_2)-\cos\pi(y_1-y_2)} - \frac{\sin\pi(y_1+y_2)}{\mathrm{ch}\,\pi(x_1-x_2)+\cos\pi(y_1+y_2)}\right];$$

$$\dot{y}_1 = \pm\frac{\gamma_2}{4}\left[\frac{\mathrm{sh}\,\pi(x_1-x_2)}{\mathrm{ch}\,\pi(x_1-x_2)-\cos\pi(y_1-y_2)} - \frac{\mathrm{sh}\,\pi(x_1-x_2)}{\mathrm{ch}\,\pi(x_1-x_2)+\cos\pi(y_1+y_2)}\right].$$

We introduce the notation

$$Q(y_1, y_2) = \exp\left(-\frac{4\pi E}{\gamma_1\gamma_2}\right)\left(\frac{2\cos\pi y_1}{\pi}\right)^{\nu}\left(\frac{2\cos\pi y_2}{\pi}\right)^{1/\nu},$$

where $\nu = \gamma_1/\gamma_2$, and E is the interaction energy. The system of dynamical equations can now be written in the form

$$\dot{x}_1 = -\frac{\gamma_1}{4}\,\mathrm{tg}\,\pi y_1 - \frac{\gamma_2(1-Q)}{4Q}\,\frac{(\sin\pi(y_1-y_2)+Q\sin\pi(y_1+y_2))}{(\cos\pi(y_1-y_2)+\cos\pi(y_1+y_2))}; \qquad (3.2)$$

$$\dot{y}_1 = \pm\frac{\gamma_2}{4}\,\frac{(1-Q)}{Q(\cos\pi(y_1-y_2)+\cos\pi(y_1+y_2))}\,\sqrt{4Q\cos^2\pi y_2-(\sin\pi(y_1-y_2)+Q\sin\pi(y_1+y_2))^2}. \qquad (3.3)$$

The equation of motion of the second vortex are obtained by mutual commutation of the subscripts 1 and 2.

The impulse P and the equation for the difference $x_1 - x_2$ are the following

$$P = \gamma_1 y_1 + \gamma_2 y_2; \qquad (3.4)$$

$$\mathrm{ch}\,\pi(x_1-x_2) = [\cos\pi(y_1-y_2)+Q\cos\pi(y_1+y_2)]/(1-Q). \qquad (3.5)$$

If y_2 from (3.4) is substituted into (3.3), the variables are separated and integration performed with respect to y_1, the solution $y_1(t)$ can be obtained in implicit form. The period of relative vortex motion will be expressed by the formula

$$T = 2\int_{y_1^*}^{y_1^{**}} \frac{4Q(\cos\pi(y_1-y_2)+\cos\pi(y_1+y_2))}{1-Q}\times \qquad (3.6)$$

$$\times\frac{dy_1}{\sqrt{4Q\cos^2\pi y_2-(\sin\pi(y_1-y_2)+Q\sin\pi(y_1+y_2))^2}},$$

where y_1^* and y_1^{**} are the "return points" determined from the condition that the radicand in (3.6) vanishes. The integral (3.6) diverges if the equality Q = 1 is satisfied somewhere in the domain of integration. The period T here becomes infinite and the motion infinite, i.e., according to (3.5) the vortices withdraw unlimitedly from each other. In the opposite case, $T < \infty$, the motion is finite and the vortices move in a bound manner. After a time T the y coordinates of the vortices will have the same values as at the beginning of the motion; the x coordinates receive an increment Δx after this time, which is easily determined by integrating (3.2) between 0 and T. It is clear that finite vortex motion will occur over closed trajectories in a coordinate system moving with the velocity $V = \Delta x/T$.

The vortex motion is determined completely by the energy E and impulse P, which depend, in turn, on the magnitude and sign of ν and the initial vortex location. There are five possible types of trajectories, examples of which are demonstrated below.

To verify all the deductions, a numerical computation program was compiled that permits solution of the system (2.5) by the Runge-Kutta method, finding the values of E and P by the given initial locations of the vortices, determination of the limits of integration y_1^* and y_1^{**} by the method of halving the segments, and calculation of the periods of the motion and

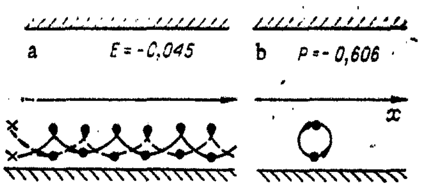

Fig. 1

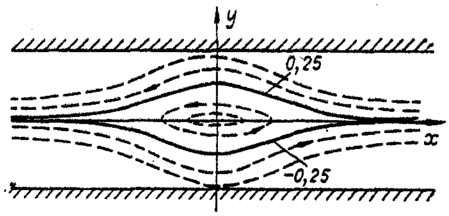

Fig. 2

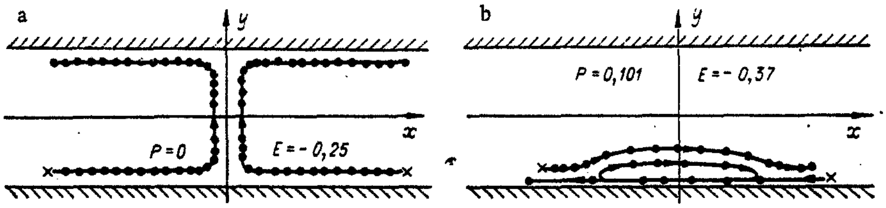

Fig. 3

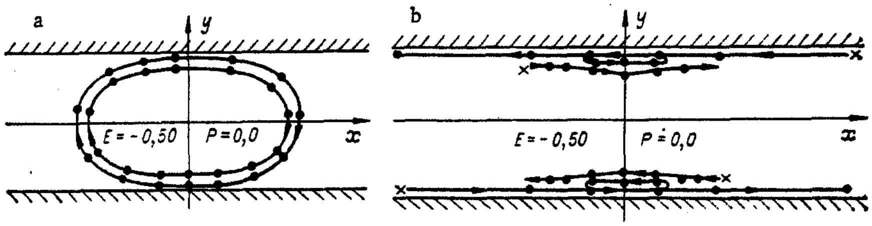

Fig. 4

the velocity of the coordinate system in which the trajectories are closed.

Periodic trajectories of the vortices $(\gamma_1 = \gamma_2 = 1, T = 0.682)$ are shown in Fig. 1 in a coordinate system at rest (a) and in a system moving at the velocity V = 0.689 (b). The points indicate the successive vortex locations; the time interval between points is t = 0.125. The initial vortex location is denoted by crosses. The separatrix dividing the finite and infinite trajectories for v = 1 and a symmetric initial vortex location is shown in Fig. 2. The trajectories of infinite motion are shown in Fig. 3 for $\nu < 0, \gamma_1 = -\gamma_2 = 1$. If the y coordinates of the initial vortex location are identical, i.e., P = 0, then the symmetric trajectory of Fig. 3a is realized. In the opposite case the trajectory of Fig. 3b is realized. The vortex closest to the wall here moves more rapidly, determining the sign of the impulse. The case when several different trajectories exist for the very same integrals of motion will be called degenerate (by analogy with quantum mechanics). The maximal degree of degeneration equals three. The trajectories of triply degenerate motion, two infinite (b) and one finite (a), are displayed in Fig. 4. Vortex motions can be classified rigorously by investigating the zeros of the radicand (3.6) for the period T.

Despite the fact that (3.2) and (3.3) are not integrated successfully in simple analytical form, all the motion characteristics can therefore be found to any degree of accuracy by numerical methods since the solution of the problem of two vortices between parallel walls is reduced to quadratures. Hence, this case can be considered as still another example of exactly integrable nonstationary ideal fluid motion.

In conclusion, we note that the theory elucidated for vortex motion is purely kinematic in nature: The Euler equations have been used. These equations can be utilized to find the pressure at any point of the flow in terms of the Cauchy–Lagrange integral.

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TWO TYPES OF VORTEX TUBE

V. A. Bubnov, A. A. Solov'ev
and Kh. Z. Ustok

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Experimental and theoretical modeling is an efficient method of solving problems of vortex generation [1-7].

It is noted that in a number of references that vertical flows in linear vortices are everywhere directed upwards, starting from the axis, right out to the boundary of the solid rotation, and only far away at the periphery does the direction change to the opposite [8, 9]. In other investiagtions the authors noted descending axial flows and flows ascending along the walls of the vertex tube [10, 11], and the vortex structure is called one-cell and two-cell, respectively. In [10] and attempt was made to classify vortex tubes, depending on the ratio of intensity of the vertical streams and the circulation. The present paper discusses an experimental study of transition from a one-cell to a two-cell vortex.

The vortex tubes are induced in a vortex chamber. The chamber height is 0.665 m. Its diameter is 0.382 m.

The upper part of the chamber has a four-blade vortex generator, mounted on the axis of a motor. The height of the rectangular blades was 0.07 m, and the width in some of the tests was 0.05 m, and in others 0.10 m. The generator was attached to the axis of the motor behind a heavy cylindrical platform. The platform diameter was 0.21 m, and its height was 0.05 m. The function of the platform was to deviate the flow, which arrived at the motor after interacting with the lower surface of the platform. In addition, this platform provided a stable frequency of rotation of the vortex generator. At the top the chamber was covered by a lid. Tests were conducted with the chamber both open and closed, and here an annular gap of the required dimensions was assigned by varying the diameter of the lid.

By using the gap in the lid we could change the dimensions of the zone in which the flow received angular momentum. In addition, as the gap changed there was a change in the

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